

The University of Chicago

A CONTRIBUTION TO
THE WARING PROBLEM FOR CUBIC
FUNCTIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1934

By
FRANCES ELLEN BAKER

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INTRODUCTION

In a recently published paper, L. E. Dickson¹ has proved the following theorems:

THEOREM 1. If $f(x) = (ax^3 + bx)/d$ ($a \neq 0$) is an integer ≥ 0 for every integer $x \geq 0$, and if $f(x) = 1$ for an integer $x > 0$, then $f(x)$ is either a pyramidal number $P(x) = (x^3 - x)/6$ or is $x + k \cdot P(x)$, where k is a positive integer. Conversely, each of the resulting functions is an integer ≥ 0 for every integer $x \geq 0$, while $f(x) = 1$ has a positive integral solution.

THEOREM 2. To each positive integer e prime to 3 there correspond positive integers C and $\nu \geq 8$ such that every integer $\geq C \cdot 3^{\nu}$ is a sum of nine values of

$$f(x) = x + e(x^3 - x)/6,$$

for integral values ≥ 0 of x .

For the case $e = 2$, he obtained a universal theorem:

THEOREM 4. Every integer ≥ 0 is a sum of nine values of $(x^3 + 2x)/3$ for integers $x \geq 0$.

In this thesis the function $f(x) = x + e(x^3 - x)/6$ is considered, where e is a multiple of 3, but not of 2. Let $e = 3a$, $a > 0$, a odd. The following theorems will be proved:

THEOREM I. To each positive odd integer a , $a \equiv 1$ modulo 3, there correspond positive integers C and $\nu \geq 8$ such that

¹L. E. Dickson, "Waring's Problem for Cubic Functions," Transactions of the American Mathematical Society, January, 1934.

every integer $\geq C \cdot 3^{37}$ is a sum of nine values of

$$f(x) = x + a(x^3 - x)/2.$$

THEOREM II. Every integer ≥ 0 is a sum of nine values of $(x^3 + x)/2$ for integers $x \geq 0$.

Methods derived from those of Dickson in the above-mentioned paper will be used in Chapter I for the proof of the first of these theorems. To prove the second, tables have been constructed showing all integers from 1 to 2733 which are sums of 2, 3, 4, 5 or 6 values of the function. Results obtained in the latter two instances have been extended to 9,530 for integers which are sums of 5 values, and to 97,079 for integers which are sums of 6 values, as explained in Chapter II. Then, by application of a well-known theorem of ascent, the proof of the second, universal theorem is completed.

CHAPTER I

PROOF OF A GENERAL THEOREM I FOR $f(x) = x + a(x^3 - x)/2$

WHEN a IS A POSITIVE, ODD INTEGER, $a \equiv 1$, MODULO 3

Consider $f(x) = x + e(x^3 - x)/6$, where the positive integer e is a multiple of 3 but not of 2. Let $e = 3a$, $a > 0$, a odd. Then the function may be written

$$(1) \quad f(x) = x + a(x^3 - x)/2.$$

In Sections 3 and 4, a will be taken congruent to 1 modulo 3; in Sections 1 and 2 it will be assumed only that $a \not\equiv 2$ modulo 3.

1. Auxiliary Lemmas. Take $f(x)$ as in (1), with $a > 0$, a odd, $a \not\equiv 2$ modulo 3.

LEMMA 1. There exists an integer m such that any given integer is congruent to $f(m)$, modulo 3^n , where n is an integer ≥ 1 .

Proof. Let

$$(2) \quad D = f(z + r) - f(z) = 3a(z^2r + zr^2)/2 + a(r^3 - r)/2 + r.$$

(a) Verification that $D \equiv 0 \pmod{3^n}$ implies $r \equiv 0 \pmod{3^n}$ when $n = 1$, and conversely.

If $D \equiv 0 \pmod{3}$, then $r \equiv 0 \pmod{3}$. Also $r \equiv 0 \pmod{3}$ implies $D \equiv 0 \pmod{3}$. Therefore D is prime to 3 if and only if r is prime to 3.

(b) Theorem. If $D \equiv 0 \pmod{3^n}$ implies $r \equiv 0 \pmod{3^n}$, then $D \equiv 0 \pmod{3^{n+1}}$ implies $r \equiv 0 \pmod{3^{n+1}}$.

For, if $D \equiv 0 \pmod{3^{n+1}}$ then $D \equiv 0 \pmod{3^n}$, which by hypothesis implies $r \equiv 0 \pmod{3^n}$, i.e. $r = 3^n R$. Then, when $n > 1$,

$$2D \equiv 3^n(2 - a) \cdot R \equiv 0 \pmod{3^{n+1}}$$

holds if and only if $R \equiv 0 \pmod{3}$, since $a \not\equiv 2 \pmod{3}$, i.e. if

$$r \equiv 0 \pmod{3^{n+1}}.$$

Therefore $D \equiv 0 \pmod{3^{n+1}}$ implies $r \equiv 0 \pmod{3^{n+1}}$ provided that

$$D \equiv 0 \pmod{3^n} \text{ implies } r \equiv 0 \pmod{3^n}.$$

For $n = 1$, the final assumption has been verified in (a). By induction, the theorem follows for $n = 2, 3, \dots$, ad inf. Thus it has been proved, for every integral $n \geq 1$, that $D \equiv 0 \pmod{3^n}$ implies $r \equiv 0 \pmod{3^n}$.

Evidently, if $r \equiv 0 \pmod{3^n}$, $D \equiv 0 \pmod{3^n}$. Hence $D \not\equiv 0 \pmod{3^n}$ if $r \not\equiv 0 \pmod{3^n}$.

Now $f(m) - f(p) = f[p + (m - p)] - f(p)$. Let $z = p$, $r = m - p$. Then $f(m) - f(p) = f(z + r) - f(z) \not\equiv 0 \pmod{3^n}$ when $0 < r < 3^n$. Hence, for $j = 0, 1, \dots, 3^n - 1$, the 3^n integers $f(j)$ are incongruent modulo 3^n , so that any integer is congruent to one of them. This completes the proof of Lemma 1.

Lemma 2 is the same as Lemma 2 of Dickson in the paper quoted above. It is restated here for reference:

LEMMA 2. If η is an odd constant integer, $v(\eta - v)$ is even, and can be made congruent to any assigned even integer modulo 2^k by choice of an integer v .

LEMMA 3. If $n > 1$ and $0 \leq m < a \cdot 3^{n+1}$, then $f(m) < \gamma \cdot 3^{3n}$, where

$$(3) \quad \gamma = 27(a^4 + 1)/2.$$

Proof. The maximum $m = 3^{n+1}a - 1$. Since $m^3 - m$ increases with m , $f(m) = m + a(m^3 - m)/2 < a \cdot 3^{n+1} + a(a^3 \cdot 3^{3n+3} - a \cdot 3^{n+1})/2 < (a^4 + 1) \cdot 3^{3n+3}/2$, if $a - a^2/2 < 3^{2n+2}/2$, which is true for all $a, n > 0$, since the maximum of $a - a^2/2$ is its value $1/2$ for $a = 1$.

Let $\mathcal{V} = 27(a^4 + 1)/2$. Then $f(m) < \mathcal{V} \cdot 3^n$, and Lemma 3 is proved.

2. Determination of Necessary Formulas. If s and C are given positive integers, a positive integer n can be chosen so that

$$C \cdot 27^n \leq s < C \cdot 27^{n+1}.$$

In the statement of Theorem I, $s \geq C \cdot 3^{3\mathcal{V}}$. Hence n can be taken so that $n \geq \mathcal{V} \geq 8$.

Any such s is one of the integers s_1 falling in the following three sub-intervals:

$$3^{1-1}C \cdot 3^{3n} \leq s_1 < 3^1C \cdot 3^{3n} \quad (i = 1, 2, 3).$$

By Lemma 1, choose an integer m_1 so that

$$s_1 = f(m_1) + 3^{nM_1}, \quad 0 \leq m_1 < 3^n,$$

where M_1 is an integer. Since $f(m_1) \geq 0$,

$$3^{nM_1} \leq s_1 < 3^1C \cdot 3^{3n}.$$

By Lemma 3,

$$3^{1-1}C \cdot 3^{3n} \leq s_1 = f(m_1) + 3^{nM_1} < \mathcal{V} \cdot 3^{3n} + 3^{nM_1}.$$

Write $M_1 = e \cdot 3^{2n} + N_1$. Then

$$(4) \quad (3^{1-1}C - \mathcal{V} - e) \cdot 3^{2n} < N_1 < (3^1C - e) \cdot 3^{2n}.$$

Take $p = 3^n$ in the identity

$$f(p - x) + f(p + x) = 2p + a(p^3 - p + 3px^2),$$

and sum for 3 values x_j of x . Thus

$$\sum_{j=1}^3 f(3^n - x_j) + f(3^n + x_j) = T,$$

where

$$T = e \cdot 3^{3n} + 3^n(eQ - e + 6), \quad Q = x_1^2 + x_2^2 + x_3^2.$$

Write φ_i for $f(v_1) + f(w_1)$ and Q_1 for Q . Then will

$$s_1 = f(m_1) + 3^n(e \cdot 3^{2n} + N_1) = f(m_1) + \varphi_i + T,$$

so that s_1 will be a sum of 9 values of $f(x)$, if

$$(5) \quad 3^n(N_1 + e - 6) = \varphi_i + e \cdot 3^n Q_1 \quad (i = 1, 2, 3).$$

Impose on the unknowns v_1, w_1 , the restriction

$$(6) \quad v_1 + w_1 = b_1 \cdot 3^{n+1} \quad (i = 1, 2, 3; b_1 \text{ a positive, odd integer}).$$

The identity

$$\varphi_i = (v_1 + w_1) \left[1 + (e/6) \{ (v_1 + w_1)^2 - 3v_1w_1 - 1 \} \right]$$

gives $\varphi_i = 3^n B_1$, where

$$(7) \quad B_1 = 3b_1 \left[1 + (e/6) \{ 9b_1^2 \cdot 3^{2n} - 1 - 3v_1(3b_1 \cdot 3^n - v_1) \} \right].$$

Insertion of $\varphi_i = 3^n B_1$ into (5) and cancellation of 3^n gives

$$(8) \quad eQ_1 = N_1 + e - 6 - B_1.$$

The v_1 will later be chosen so that

$$(9) \quad 0 \leq v_1 \leq 3b_1 \cdot 3^n, \quad 0 \leq N_1 + e - 6 - B_1 \leq e \cdot 3^{2n}.$$

These and (6) and (8) imply

$$(10) \quad 0 \leq w_1, \quad 0 \leq Q_1 \leq 3^{2n}.$$

It will later be proved that Q_1 may be taken as a sum of three squares of integers $x_j \geq 0$. Thus $x_j \leq 3^n$ by (10). It will then follow that s_1 is the sum of the values of $f(x)$ for the nine values $m_1, v_1, w_1, 3^n - x_j, 3^n + x_j$ of x , each an integer ≥ 0 .

Employ the abbreviation

$$(11) \quad V_1 = v_1 - 3b_1 \cdot 3^n/2.$$

In the right member of (8) insert the value of B_1 , and get

$$s_1 = N_1 + e - 6 - 3b_1[1 + (e/6)(3V_1^2 + 9b_1^2 \cdot 3^{2n/4} - 1)].$$

The final condition (9) is $0 \leq s_1 \leq e \cdot 3^{2n}$. Now $s_1 \geq 0$ if $A_1/3 \geq V_1^2$, where

$$(12) \quad A_1 = \frac{6}{e} \left[\frac{N_1 + e - 6}{3b_1} - 1 \right] - \frac{9}{4} b_1^2 \cdot 3^{2n} + 1.$$

Hence $s_1 \geq 0$ if

$$(13) \quad A_1 \geq 0, \quad V_1 \geq 0, \quad \sqrt{A_1/3} \geq V_1.$$

Next, $s_1 \leq e \cdot 3^{2n}$ if $G_1/3 \leq V_1^2$, where

$$(14) \quad G_1 = A_1 - 2 \cdot 3^{2n}/b_1,$$

and hence if

$$(15) \quad G_1 \geq 0, \quad V_1 \geq 0, \quad \sqrt{G_1/3} \leq V_1.$$

If it is assumed that $G_1 \geq 0$ (whence $A_1 \geq 0$), as well as

$$(16) \quad 3b_1 \cdot 3^n/2 + \sqrt{G_1/3} \leq v_1 \leq 3b_1 \cdot 3^n/2 + \sqrt{A_1/3}, \quad \sqrt{A_1/3} \leq 3b_1 \cdot 3^n/2,$$

it is seen that (13) and (14) follow, and that $v_1 \leq 3b_1 \cdot 3^n$,

whence (9) hold.

The use of the values (12) and (14) of A_1 and G_1 shows that the condition $G_1 \geq 0$ and the final inequality (16) are equivalent to

$$\begin{aligned} L_1 &\leq N_1 \leq U_1, & L_1 &= e \cdot 3^{2n} + 9e \cdot b_1^3 \cdot 3^{2n/8} + \beta_i, \\ & & U_1 &= 9e \cdot b_1^3 \cdot 3^{2n/2} + \beta_i, \end{aligned}$$

where

$$(17) \quad \beta_i \equiv b_i(3 - e/2) + 6 - e = (1 + b_i/2)(6 - e).$$

This inequality will evidently follow if L_i is $\leq$ the lower limit in (4) and if U_i is $\geq$ the upper limit in (4), and hence if

$$(18) \quad 2e + \mathcal{V} + 9eb_1^3/8 + \beta_1/3^{2n} \leq 3^{1-L_i} \leq 3eb_1^3/2 + e/3 + \beta_1/3^{2n+1} \\ (i = 1, 2, 3).$$

Since it will be necessary to assign to v_1 a prescribed residue modulo 8, at least 8 consecutive integral values of v_1 must be found to satisfy the first inequality (16) for every $i = 1, 2, 3$. The difference between its limits is

$$K_1 = \sqrt{A_1/3} - \sqrt{G_1/3}.$$

Write $g_1 = 2 \cdot 3^{2n}/(b_1 A_1)$. By (14) and (15), $0 < g_1 \leq 1$. Thus K_1 is the product of $\sqrt{A_1/3}$ by

$$1 - \sqrt{1 - g_1} = g_1/(1 + \sqrt{1 - g_1}) > g_1/2.$$

Hence

$$(19) \quad K_1 > 3^{2n}/b_1 \sqrt{3A_1}.$$

By (4) and (12),

$$(20) \quad 3A_1 < \frac{18}{e} \left[\frac{(3^{1-L_i} - e)3^{2n} + e - 6}{3b_1} - 1 \right] - \frac{27}{4} b_1^2 3^{2n} + 3.$$

3. Proof of Theorem I when $e = 3$, $a = 1$. When $e = 3$, $n \geq 8$, inequalities (18) all hold if

$$(21) \quad b_1 = 5, \quad b_2 = 7, \quad b_3 = 11, \quad \mathcal{V} = 27.$$

Then

$$\begin{aligned} 454.8 &\leq C \leq 563.5, \\ 1190.6 &\leq 3C \leq 1544.5, \\ 4525.1 &\leq 9C \leq 5990.5, \end{aligned}$$

which allow $C = 504$. Also, by (21), (19) and (20),

$$\begin{aligned} 3A_1 &< 435 \cdot 3^{2n} < 21^2 \cdot 3^{2n}, & K_1 &> 62, \\ 3A_2 &< 965 \cdot 3^{2n} < 32^2 \cdot 3^{2n}, & K_2 &> 29, \\ 3A_3 &< 1657 \cdot 3^{2n} < 41^2 \cdot 3^{2n}, & K_3 &> 14, \end{aligned}$$

whence each $K_i > 8$. By (7)

$$(22) \quad 2B_1 - 6b_1 = b_1 e F,$$

where F denotes the quantity in braces in (7). By Lemma 2, v_1 can be chosen modulo 8 so that F is congruent modulo 8 to any assigned even integer. Thus in (8), v_1 can be chosen modulo 8 so that $2eQ_1 \equiv 2z \pmod{8}$, where z is an arbitrary integer. Take $z = e = 3$. Then $Q_1 \equiv 1 \pmod{4}$. But $Q_1 > 0$. Hence Q_1 is a sum of three integral squares. This proves Theorem I when $e = 3$, with $C = 504$, $\nu = 8$.

4. Proof of Theorem I when $e = 3a$, $a \equiv 1 \pmod{3}$. The values of b_1, b_2, b_3, C, ν , will first be determined so that all these inequalities (18) hold when $n \geq \nu$, viz.

$$(23) \quad I_1 \leq 3^{i-1} C \leq S_1 \quad (i = 1, 2, 3).$$

Minimum values of C and b_1 may be found by the following scheme. Take b_1 to be the least positive odd integer for which $I_1 \leq S_1$. Take b_2 and b_3 to be positive odd integers for which $S_2 \geq 3C, S_3 \geq 9C$. The tablette displays the results for $a = 1, 7, 13, 19, 25$.

p	a = 6p + 1	e = 3a	b ₁	b ₂	b ₃	C	$\mathcal{V} = 27(a^4 + 1)/2$
0	1	3	5	7	11	504	27
1	7	21	17	25	35	148,540	32,427
2	13	39	31	45	65	1,692,747	385,587
3	19	57	45	67	93	7,602,854	1,759,347
4	25	75	59	87	123	22,602,457	5,273,451

For these values, $I_2 \leq 3C$, $I_3 \leq 9C$, whence (23) hold.

For a general a, choose b_1 to be a linear function of p which has the value in the tablette when $a = 7, \dots, 25$, and is such that (23) hold as regards the coefficients of the highest power of p.

With $a = 6p + 1$, take $b_1 = 14p + 3$, actually the least odd b_1 when $p = 1, 2, 3$ or 4. By (18),

$$C \geq 2e + \mathcal{V} + 9eb_1^3/8 + \beta_1/3^{2n},$$

$$C > 2(18p + 3) + 27[(6p + 1)^4 + 1]/2 + 9(18p + 3)(14p + 3)^3/8,$$

whence C may be taken as

$$C = 73,062p^4 + 56,646p^3 + 16,524p^2 + 2183p + 125.$$

Then $I_1 < C < S_1$.

(1) If p is odd, take $b_2 = 21p + 4$. Computation gives

$$\begin{aligned} I_2 &< 205,031.25p^4 + 150,082.875p^3 + 41,188.5p^2 + 5058p + 250 \\ &< 3C, \end{aligned}$$

termwise (as to coefficients of powers of p).

$$S_2 > 250,047p^4 + 184,558.5p^3 + 51,029p^2 + 6269p + 289 > 3C,$$

termwise except for the linear and constant terms; for $p \geq 1$,

$S_2 > 3C$. Take $b_3 = 29p + 6$. By computation,

$$I_3 < 511,373.25p^4 + 400,521.375p^3 + 117,429.75p^2 + 15,304.5p + 763 < 9C,$$

termwise.

$$S_3 > 658,503p^4 + 518,476.5p^3 + 152,684p^2 + 19,931p + 973 > 9C,$$

termwise except for the constant; for $p \geq 1$, $S_3 > 9C$.

(2) If p is even, > 0 , take $b_2 = 21p + 3$, $b_3 = 29p + 7$.

Since these b_3 exceed the b_3 in case (1), evidently $S_3 > 9C$.

Similarly, these b_2 are less than the b_2 of case (1), so $I_2 < 3C$.

Computation gives

$$S_2 > 250,047p^4 + 148,837.5p^3 + 33,169p^2 + 3286p + 122.5 > 3C,$$

since $p \geq 2$.

$$I_3 < 511,373.25p^4 + 451,612.125p^3 + 148,847.625p^2 + 21,693.375p + 1191 < 9C$$

since $p \geq 2$.

Thus all the inequalities (23) hold with the general b_1 found above.

From (20) and the square of (19) it follows that $K_1 > 8$ if 3^{2n} exceeds a certain function of p and i , and hence if n is sufficiently large. Then there exist more than 8 consecutive integral values of v_1 which satisfy inequalities (16).

To prove that Q_1 is an integer, Lemma 1 is employed with n replaced by $n + 1$. If s_1 is any given integer, this lemma shows the existence of an integer t_1 such that

$$s_1 = f(t_1) + 3 \cdot 3^n u_1, \quad 0 \leq t_1 < 3^{n+1}$$

where u_1 is an integer. In (2), take $z = t_1$, $r = 3^{n+1} y_1$, where

y_1 is an arbitrary integer. Then

$$f(t_1 + 3^{n+1}y_1) - f(t_1) = D = r + a(3z^2r + 3zr^2 + r^3 - r)/2.$$

Since $3z^2r + 3zr^2 + r^3 - r$ is a multiple of 3 and 2,

$$D \equiv r \pmod{3a}.$$

Also, $2D$ has the factor r and therefore the factor 3^{n+1} , so

$D = 3^{n+1}E$, E an integer. Therefore

$$3^{n+1}E \equiv r \equiv 3^{n+1}y_1 \pmod{3a}.$$

Since a is prime to 3^n , $E \equiv y_1 \pmod{a}$. Write

$$m_1 = t_1 + 3^{n+1}y_1, \quad q_1 = u_1 - E.$$

Then

$$f(m_1) - f(t_1) = D = 3^{n+1}E.$$

Thus

$$s_1 = f(t_1) + 3^{n+1}(q_1 + E) = f(m_1) + 3^{n+1}q_1.$$

Write $3q_1 = M_1$. Then

$$s_1 = f(m_1) + 3^n M_1.$$

Since a is prime to 3, an integer y_1 can be chosen so that

$$(24) \quad y_1 \equiv u_1 - (2 + b_1) \pmod{a}, \quad 0 \leq y_1 < a,$$

where b_1 is the above positive, odd integer, and u_1 is the integer found above. Thus y_1 is completely determined. The inequality of (24) shows that the maximum m_1 is $3^{n+1} - 1 + 3^{n+1}(a - 1) = a \cdot 3^{n+1} - 1$. Hence $0 \leq m_1 < a \cdot 3^n$. In the congruence of (24), replace y_1 by $E \pmod{a}$, and $u_1 - E$ by $q_1 \pmod{a}$. Therefore

$$q_1 - 2 - b_1 \equiv 0 \pmod{a}.$$

Multiply by 3, and $3q_1 - 6 - 3b_1 \equiv 0 \pmod{3a}$. Since $e = 3a$, and $3q_1 = M_1$, this congruence becomes

$$M_1 - 6 - 3b_1 \equiv 0 \pmod{e}.$$

As before, write $M_1 = e \cdot 3^{2n} + N_1$, i.e. $M_1 \equiv N_1 \pmod{e}$.

Then

$$N_1 - 6 - 3b_1 \equiv 0 \pmod{e}$$

$$\text{and} \quad N_1 + e - 6 - 3b_1 \equiv 0 \pmod{e}.$$

Since b_1 is odd, the quantity in braces in (7) is even, whence $B_1 \equiv 3b_1 \pmod{e}$. Therefore

$$N_1 + e - 6 - B_1 \equiv 0 \pmod{e},$$

whence (8) gives an integral Q_1 , as desired.

Moreover, $Q_1 \equiv 1 \pmod{4}$. For, it was proved that there exist more than 8 consecutive integral values of v_1 which satisfy inequalities (16). By (7),

$$2B_1 \equiv 6b_1 - 3b_1 e v_1 (\eta - v_1) \pmod{8}, \quad \eta = 3b_1 \cdot 3^n.$$

By Lemma 2, v_1 can be chosen modulo 8 so that $v_1 (\eta - v_1) \equiv 2 \zeta_i \pmod{8}$, where ζ_i is an assigned integer. Then

$$N_1 + e - 6 - B_1 \equiv N_1 + e - 6 - 3b_1 + 3b_1 e \zeta_i \pmod{4}.$$

Choose ζ_i so that the second member is congruent to e , modulo 4. Then (8) gives $Q_1 \equiv 1 \pmod{4}$.

Since inequalities (18) were satisfied at the outset, it is known that $G_1 \geq 0$ and that the final inequality (16) holds. Also, (16₁) was shown to hold. Hence (9) hold. By (9₂), $0 \leq eQ_1 \leq e \cdot 3^{2n}$. Since Q_1 is an integer ≥ 0 such that $Q_1 \equiv 1$

(mod 4), it is well known that

$$Q_1 = \sum_{j=1}^3 x_j^2,$$

where the x_j are integers ≥ 0 . But $Q_1 \leq 3^{2n}$, whence each $x_j \leq 3^n$.

In view of (9_1) , the integer w_1 defined by (6) is ≥ 0 . Then $f(v_1) + f(w_1) \equiv \phi_i$ has the value $3^n B_1$. Hence (8) yields (5), which is the condition that s_1 be the sum of the values of $f(x)$ for the nine values $m_1, v_1, w_1, 3^n - x_j, 3^n + x_j$, of x , each an integer ≥ 0 . This proves

THEOREM I. To each positive, odd integer a , $a \equiv 1$ modulo 3, there correspond positive integers C and $\nu \geq 8$ such that every integer $\geq C \cdot 3^{3\nu}$ is a sum of nine values of

$$f(x) = x + a(x^3 - x)/2.$$

CHAPTER II

PROOF OF A UNIVERSAL THEOREM II FOR $f(x) = x + (x^3 - x)/2$

When $e = 3$, $a = 1$, it was shown that (18) are satisfied if $b_1 = 5$, $b_2 = 7$, $b_3 = 11$, $C = 504$. Take $n \geq 1' \geq 8$, $K_1 > 62$, $K_2 > 29$, $K_3 > 14$, as in Section 3. Then Theorem I shows that every integer $\geq 504 \cdot 3^{24}$ is a sum of nine values of

$$(25) \quad f(x) = x + (x^3 - x)/2 = (x^3 + x)/2$$

for integral values ≥ 0 of x . To show that every integer $< 504 \cdot 3^{24}$ is a sum of nine values of $f(x)$, use is made of tables whose essential results are summarized below.

5. Summary of Tables Constructed for the Interval 1 to 2733. Tables were made showing all integers 1 to 2733 which are sums of 2, 3, 4, 5 or 6 values of $f(x)$. In particular, with the three exceptions 29, 58 and 63, all integers 1 to 2733 are sums of 6 values of the function. Moreover, all integers 1 to 2733, with the following 69 exceptions are sums of 5 values.

(a) List of Integers 1 to 2733 Requiring More

Than 5 Functional Values

14, 24, 28, 29, 43, 48, 53, 57, 58, 62, 63, 92, 154, 158, 159, 173, 188, 202, 204, 208, 217, 231, 236, 269, 283, 298, 312, 318, 337, 347, 383, 422, 426, 493, 494, 528, 532, 543, 557, 558, 602, 628, 642, 667, 713, 828, 841, 858, 897, 922, 964, 1067, 1082, 1100, 1138, 1163, 1228, 1298, 1364, 1402, 1553, 1596, 1648, 1663, 1694, 1838, 1968, 2012, 2658.

All integers 1 to 2733, with 786 exceptions are sums of 4 values.

(b) List of Integers in the Intervals 1 - 1862* and 2607 - 2733
Requiring More Than 4 Values of $f(x) = (x^3 + x)/2$

9	168	307	463	602	793	962	1099	1263	1421	1599	1817
13	70	11	67	03	94	63	1100	64	26	1614	23
14	71	12	72	08	807	64	01	66	27	18	31
19	72	13	75	13	11	67	02	68	32	27	33
23	73	15	77	20	13	68	03	72	34	29	34
24	74	17	78	23	15	71	04	76	39	31	37
27	83	18	79	27	20	75	17	79	42	32	38
28	87	19	88	28	22	77	18	83	57	33	49
29	88	22	89	33	23	88	19	89	63	37	53
33	89	23	91	37	24	89	23	91	65	38	54
38	93	32	92	41	26	92	24	93	66	43	57
42	97	36	93	42	27	93	27	94	67	46	
43	99	37	94	47	28	94	28	97	69	47	
47	201	42	98	48	29	95	29	98	71	48	2607
48	02	44	509	52	32	1002	33	99	73	58	09
52	03	46	13	53	34	06	37	1303	82	60	12
53	04	47	17	54	36	07	38	11	85	62	16
56	07	49	18	58	39	08	42	12	88	63	17
57	08	53	23	61	40	09	43	13	97	68	21
58	12	57	24	62	41	14	47	16	98	72	24
61	16	58	27	66	42	17	48	17	1507	77	28
62	17	61	28	67	43	18	58	18	11	79	32
63	18	63	31	79	44	23	62	22	19	85	36
77	21	68	32	83	53	27	63	23	21	87	37
87	26	78	37	93	57	28	67	24	27	89	38
89	30	82	38	98	58	29	83	25	28	91	39
91	31	83	42	703	59	31	84	30	31	93	43
92	33	88	43	04	63	33	87	33	34	94	48
93	34	92	52	08	65	35	89	36	35	1703	53
94	35	96	53	12	67	37	93	37	36	08	54
97	36	407	56	13	78	38	94	40	37	18	57
106	47	11	57	17	82	39	97	41	38	23	58
08	49	17	58	18	88	48	1201	49	48	24	62
20	51	21	62	22	92	52	02	53	52	27	63
23	53	22	63	23	96	53	08	59	53	28	67
24	54	25	66	26	97	58	12	63	62	32	68
25	59	26	67	30	99	62	13	64	63	33	69
37	64	27	68	31	903	66	14	68	67	37	78
39	68	28	77	33	07	67	23	73	68	43	82
43	69	29	81	47	17	69	24	74	72	47	94
44	72	32	82	49	21	71	27	78	77	52	96
49	73	43	84	57	22	72	28	83	78	57	97
52	78	46	87	62	25	73	33	87	81	67	98
53	82	47	89	63	29	77	34	88	83	71	99
54	83	48	94	64	30	81	37	93	91	72	2701
57	84	53	95	69	48	82	47	97	92	73	03
58	93	56	97	76	49	85	52	1401	94	88	04
59	97	57	98	78	52	87	53	02	95	93	09
62	98	58	99	86	56	91	57	03	96	98	12
63	303	59	601	89	59	95	61	17	98	1803	14
66	60			90		98		20		04	18
											23

*Note: For the interval 1863-2606 see Section 6 (b), Case I. Only the portion 1775-2606, used in the proof of Theorem II, has been checked more than once.

6. Proof that All Integers from 2734 to 9530, with the Exceptions 2978, 3353, 3372 and 8107, are Sums of 5 Functional Values. The integers 2734 to 9530 were proved to be sums of 5 values by a device of Dickson used in the paper previously cited. A list was made of the integers in the interval 1775-2606 which are missing from the table of sums by 4. This interval was chosen because (1) it has relatively fewer integers of this kind than do other intervals of the range 1-2733; (2) it lies near the upper end of the table. To each of those integers requiring a sum of 5 values and lying, for example, in the interval 1863 to 2606, is added $870 = f(12)$, and from the sum is subtracted $1105 = f(13)$. (Actually, $1105 - 870 = 235$ is subtracted from each integer in the list.) If the resulting difference is a sum of 4 values, evidently the original integer plus 870 is a sum of 5 values. If the difference does not lie in the table of sums by 4, other values of $f(x)$ may be subtracted from the integer plus 870, provided only that the remainders fall within the range 1-2733. Obviously, those integers in the interval 1863 to 2606 which are sums of 4 values will automatically extend to a sum of 5 by the addition of a functional value, and need not be written down. Thus the table for sums by 5 has been extended for all integers from $1863 + 870 = 2733$ to $2606 + 870 = 3476$.

In the further application of this process, functional values up to $f(27) = 8801$, and certain of their differences, are used. For reference, a list of these is appended.

(a) Values of $f(x) = (x^3 + x)/2$ up to $f(27) = 8801$

and Their Needed Differences

The first twelve values of $f(x)$ are, in order, 0, 1, 5, 15, 34, 65, 111, 175, 260, 369, 505 and 671. Succeeding values, beginning with 870, are displayed horizontally up to 6924, vertically up to 8801.

$f(x)$	870	1105	1379	1695	2056	2465	2925
671	+199						
870	0						
1105	-235	0					
1379	-509	-274	0	+316			
1695	-825	-590	-316	0			
2056	-1186	-951	-677	-361	0	+409	
2465	-1595	-1360	-1086	-770	-409	0	+460
2925	-2055	-1820	-1546	-1230	-869	-460	0
3439		-2334	-2060	-1744	-1383	-974	-514
4010				-2315	-1954	-1545	-1085
4641						-2176	-1716
5335							-2410
$f(x)$	3439	4010	4641	5335	6095	6924	
2925	+514						
3439	0	+571					
4010	-571	0	+631				
4641	-1202	-631	0	+694			
5335	-1896	-1325	-694	0	+760		
6095		-2085	-1454	-760	0	+829	
6924			-2283	-1589	-829	0	
7825				-2490	-1730	-901	
8801						-1877	

(b) Proof for Thirteen Subdivisions of the Interval 2734 to 9530

Case I. PROOF THAT ALL INTEGERS FROM $1863 + 870 = 2733$ TO $2606 + 870 = 3476$, WITH THE EXCEPTIONS 2978, 3353 AND 3372, ARE SUMS OF 5 VALUES OF $f(x)$.

The integers occurring in the interval 1863 to 2606 which require more than 4 values of $f(x)$ are listed in column A. Corresponding to a certain integer in A, entries of remainders (obtained as described above) are made in the succeeding columns to the right, concluding with the first one reached which can be expressed as a sum of 4 functional values. In case the final remainder still requires more than four values, this fact is indicated by placing a bracket after the last entry. Further examination of the sum of the original integer plus the additive functional value is then necessary. Resultant exceptions in the list of integers expressible as a sum of 5 values are indicated in the title of the subdivision.

A	-235	-509	-825	-1186	-1595	-2055	+199
1863	1628						
1865	1630						
1867	1632	1358					
1869	1634						
1873	1638	1364	1048	687			
1882	1647	1373	1057				
1883	1648	1374	1058	697			
1897	1662	1388	1072	711			
1898	1663	1389					
1901	1666						
1903	1668	1394					
1907	1672	1398					
1913	1678						
1926	1691	1417	1101	740			
1927	1692						
1929	1694	1420	1104	743			
1931	1696						
1934	1699						
1937	1702						
1939	1704						
1942	1707						
1943	1708	1434	1118	757	348		

Case I (Continued)

A	-235	-509	-825	-1186	-1595	-2055	+199
1946	1711						
1947	1712						
1948	1713						
1953	1718	1444					
1959	1724	1450					
1963	1728	1454					
1967	1732	1458					
1968	1733	1459					
1972	1737	1463	1147	786	377		
1973	1738						
1974	1739						
1978	1743	1469	1153				
1987	1752	1478					
1992	1757	1483					
1993	1758						
1997	1762						
1999	1764						
2002	1767	1493					
2003	1768						
2006	1771	1497	1181				
2007	1772	1498	1182				
2011	1776						
2012	1777						
2017	1782						
2019	1784						
2027	1792						
2030	1795						
2031	1796						
2032	1797						
2033	1798	1524					
2038	1803	1529					
2039	1804	1530					
2048	1813						
2049	1814						
2053	1818						
2068	1833	1559					
2075	1840						
2082	1847						
2083	1848						
2088	1853	1579					
2097	1862						
2102	1867	1593					
2104	1869	1595	1279	918			
2108	1873	1599	1283	922	513	53	2307]
2112	1877						
2128	1893						
2133	1898	1624					
2138	1903	1629	1313	952	543	83	
2142	1907	1633	1317	956	547		
2148	1913	1639					
2152	1917						
2153	1918						
2171	1936						

Case I (Continued)

A	-235	-509	-825	-1186	-1595	-2055	+199
2181	1946	1672	1356				
2185	1950						
2192	1957						
2193	1958						
2196	1961						
2198	1963	1689	1373	1012			
2199	1964						
2203	1968	1694	1378	1017	608	148	
2204	1969						
2207	1972	1698					
2208	1973	1699					
2214	1979						
2218	1983						
2219	1984						
2222	1987	1713					
2223	1988						
2224	1989						
2229	1994						
2238	2003	1729					
2242	2007	1733	1417	1056			
2243	2008						
2248	2013						
2252	2017	1743	1427	1066	657		
2257	2022						
2262	2027	1753					
2263	2028						
2267	2032	1758					
2271	2036						
2274	2039	1765					
2285	2050						
2289	2054						
2291	2056						
2294	2059						
2295	2060						
2302	2067						
2303	2068	1794					
2307	2072						
2308	2073						
2313	2078						
2327	2092						
2328	2093						
2333	2098						
2334	2099						
2335	2100						
2337	2102	1828					
2338	2103						
2341	2106						
2353	2118						
2354	2119						
2362	2127						
2363	2128	1854	1538	1177			
2370	2135						
2373	2138	1864					

Case I (Continued)

A	-235	-509	-825	-1186	-1595	-2055	+199
2374	2139						
2383	2148	1874					
2388	2153	1879					
2391	2156						
2393	2158						
2395	2160						
2397	2162						
2398	2163						
2399	2164						
2402	2167						
2403	2168						
2408	2173						
2413	2178						
2417	2182						
2418	2183						
2422	2187						
2437	2202						
2443	2208	1934	1618	1257	848		
2444	2209						
2449	2214	1940					
2451	2216						
2457	2222	1948	1632	1271			
2463	2228						
2469	2234						
2473	2238	1964					
2479	2244						
2483	2248	1974	1658	1297	888	428	2682]
2487	2252	1978	1662	1301			
2488	2253						
2497	2262	1988					
2502	2267	1993	1677	1316	907	447	2701]
2503	2268						
2507	2272						
2508	2273						
2516	2281						
2520	2285	2011	1695				
2521	2286						
2526	2291	2017	1701				
2527	2292						
2528	2293						
2539	2304						
2547	2312						
2553	2318						
2557	2322						
2559	2324						
2568	2333	2059					
2572	2337	2063					
2573	2338	2064					
2587	2352						
2590	2355						
2593	2358						
2597	2362	2088	1772	1411			

Case II. PROOF THAT ALL INTEGERS FROM $2370 + 1105 = 3475$ TO $2606 + 1105 = 3711$ ARE SUMS OF 5 VALUES OF $f(x)$.

The work for Cases II to XIII appears in condensed form. In Case II, for example, 274 is subtracted from the integers of column A, which covers the interval 2370-2606 as listed in A of Case I, and does not appear here. The subtraction shows that all these integers are sums of 4 values, except the ones entered in the column headed -274. Next, the subtraction of 590 from the integers of the previous column disposes of all except those entered in the column headed -590. The process continues until the subtraction of 2334 has eliminated all the integers of A.

-274	-590	-951	-1360	-1820	-2334
2402					
2422					
2473	2473				
2488	2488	2488	2488		
2497	2497				
2503	2503	2503	2503	2503	
2516	2516				
2526					
2559					
2568	2568				
2587	2587				

Case III. PROOF THAT ALL INTEGERS FROM $2333 + 1379 = 3712$ TO $2606 + 1379 = 3985$ ARE SUMS OF 5 VALUES OF $f(x)$.

-316	-316	-677	-1086	-1546	-2060
2333	2487	2335			
2335	2497	2354	2354		
2354	2508	2444			
2391	2520	2449	2449	2449	
2398	2539	2508			
2399	2559	2539			
2413	2568	2559	2559		
2418	2573	2590			
2444	2587				
2449	2590				
2469					

Case IV. PROOF THAT ALL INTEGERS FROM $2291 + 1695 = 3986$
TO $2606 + 1695 = 4301$ ARE SUMS OF 5 VALUES OF $f(x)$.

-361 ¹	-361	-770	-1230	-1744	-2315	+316
2295	2393	2307	2307	2307	2307	
2303	2399	2308				
2307	2443	2333	2333	2333		
2308	2444	2353	2353			
2328	2449	2388	2388			
2333	2463	2399				
2334	2469	2449				
2335	2473	2463	2463			
2353	2503	2473				
2354	2553	2503				
2363	2557	2568				
2373	2559					
2388	2568					
2391	2590					

Case V. PROOF THAT ALL INTEGERS FROM $2243 + 2056 = 4299$
TO $2606 + 2056 = 4662$ ARE SUMS OF 5 VALUES OF $f(x)$.

-409	-409	-869	-1383	-1954
2243	2383	2243		
2262	2402	2262		
2263	2408	2335	2335	
2274	2457	2338		
2291	2497			
2307	2521			
2335	2547			
2338	2557			
2362	2590			

Case VI. PROOF THAT ALL INTEGERS FROM $2198 + 2465 = 4663$
TO $2606 + 2465 = 5071$ ARE SUMS OF 5 VALUES OF $f(x)$.

-460	-460	-460	-974	-974	-1545	-2176	+409
2203	2363	2463	2207	2568	2207		
2207	2373	2479	2263	2572	2263	2263	
2248	2391	2487	2291		2408		
2263	2397	2502	2327		2508	2508	2508
2291	2399	2508	2333		2568	2568	2568
2294	2402	2528	2391		2572	2572	2572
2295	2403	2557	2408				
2313	2408	2568	2413				
2327	2413	2572	2508				
2333	2457	2593	2557				

Apparent exceptions are in fact sums of 5 values:

$$4973 = 15 + 175 + 671 + 2056 + 2056$$

$$5033 = 65 + 1105 + 1105 + 1379 + 1379$$

$$5037 = 260 + 671 + 671 + 1379 + 2056$$

Case VII. PROOF THAT ALL INTEGERS FROM $2142 + 2925 = 5067$
TO $2606 + 2925 = 5531$ ARE SUMS OF 5 VALUES OF $f(x)$.

-514	-514	-514	-1085	-1716	-2410	+460
2152	2285	2473	2152			
2193	2302	2487	2203			
2199	2307	2488	2208	2208	2208	2208
2203	2337	2507	2222			
2207	2363	2516	2337			
2208	2383	2520	2383	2383	2383	2383
2222	2397	2521	2397			
2238	2417	2526	2473	2473	2473	2473
2242	2443	2547	2487			
2257	2451	2553	2488			
2271	2457	2597				

Apparent exceptions are in fact sums of 5 values:

$$5133 = 175 + 505 + 1379 + 1379 + 1695$$

$$5308 = 5 + 111 + 671 + 2056 + 2465$$

$$5398 = 34 + 65 + 369 + 2465 + 2465$$

Case VIII. PROOF THAT ALL INTEGERS FROM $2088 + 3439 = 5527$ TO $2606 + 3439 = 6045$ ARE SUMS OF 5 VALUES OF $f(x)$.

-571	-571	-1202	-1896	+514
2102	2294	2204		
2108	2295	2208	2208	
2133	2303	2218		
2138	2308	2229	2229	2229
2148	2328	2274	2274	2274
2152	2338	2289		
2185	2374	2303	2303	2303
2198	2388	2539		
2203	2402	2590		
2204	2408			
2208	2444			
2214	2469			
2218	2497			
2219	2502			
2229	2508			
2243	2539			
2248	2568			
2262	2573			
2274	2590			
2289				

Apparent exceptions are in fact sums of 5 values:

$$5668 = 0 + 111 + 111 + 111 + 5335$$

$$5713 = 15 + 15 + 1379 + 1379 + 2925$$

$$5742 = 5 + 111 + 1105 + 2056 + 2465$$

Case IX. PROOF THAT ALL INTEGERS FROM $2033 + 4010 = 6043$ TO $2606 + 4010 = 6616$ ARE SUMS OF 5 VALUES OF $f(x)$.

-631	-631	-1325	-2085	+571	E*
2048	2289	2048	2048		
2088	2291	2088			
2097	2294	2138	2138	2138	6148
2102	2303	2152			
2104	2308	2203			
2128	2334	2222	2222	2222	6232
2138	2354	2274	2274	2274	6284
2142	2363	2289	2289	2289	6299
2152	2374	2334	2334	2334	6344
2193	2383	2354	2354	2354	6364
2198	2388	2363	2363	2363	6373
2199	2398	2383	2383	2383	6393
2203	2402	2398	2398	2398	6408
2208	2403	2402	2402	2402	6412
2214	2469	2488			
2222	2488				
2223	2528				
2229	2557				
2262	2568				
2263	2573				
2274	2590				

*Column E in Cases IX-XIII contains apparent exceptions which, like those of Cases VI-VIII, have been shown to be sums of 5 values.

Case X. PROOF THAT ALL INTEGERS FROM $1974 + 4641 = 6615$
TO $2606 + 4641 = 7247$ ARE SUMS OF 5 VALUES OF $f(x)$.

-694	-694	-1454	-2283	+631	E
1987	2285	1992	1992		
1992	2289	1997	1997	1997	6638
1993	2308	2006	2006	2006	6647
1997	2327	2007	2007	2007	6648
2006	2337	2011	2011		
2007	2341	2012	2012	2012	6653
2011	2354	2017	2017	2017	6658
2012	2362	2031	2031	2031	6672
2017	2373	2053	2053		
2019	2383	2082	2082		
2027	2388	2133	2133	2133	6774
2030	2397	2383			
2031	2402	2402			
2053	2417	2417			
2068	2418	2418			
2082	2422	2422	2422	2422	7063
2097	2437	2487	2487	2487	7128
2128	2451	2527			
2133	2487	2557			
2192	2497	2597			
2222	2527				
2229	2547				
2242	2557				
2257	2559				
2262	2597				
2271					

Case XI. PROOF THAT ALL INTEGERS FROM $1913 + 5335 = 7248$

TO $2606 + 5335 = 7941$ ARE SUMS OF 5 VALUES OF $f(x)$.

-760	-760	-1589	-2490	+694	E
1927	2291	1947	1947		
1943	2294	1972	1972		
1947	2295	2017	2017		
1953	2308	2032	2032		
1968	2313	2049	2049	2049	7384
1972	2327	2082	2082	2082	7417
1973	2328	2083	2083	2083	7418
1974	2337	2192	2192	2192	7527
1987	2338	2242	2242	2242	7577
1993	2341	2338	2338	2338	7673
1997	2354	2418	2418	2418	7753
2007	2374	2488	2488	2488	7823
2012	2391	2553	2553	2553	7888
2017	2393	2597			
2032	2397				
2039	2398				
2049	2403				
2053	2408				
2082	2418				
2083	2422				
2097	2437				
2128	2449				
2133	2451				
2138	2463				
2148	2483				
2153	2487				
2181	2488				
2192	2497				
2199	2503				
2223	2507				
2229	2527				
2242	2553				
2248	2593				
2257	2597				
2267					
2271					

Case XII. PROOF THAT ALL INTEGERS FROM $1838 + 6095 = 7933$ TO $2606 + 6095 = 8701$, WITH THE EXCEPTION 8107, ARE SUMS OF 5 VALUES OF $f(x)$.

-829	-829	-1730	+760	E
1838	2152	1838		
1857	2153	1867		
1867	2192	1882		
1882	2193	1898	1898	7993
1898	2203	1901		
1901	2207	1927		
1927	2222	1929		
1929	2263	1931		
1931	2271	1946		
1946	2294	1947		
1947	2295	1948		
1948	2302	2012	2012	8107*
1953	2327	2053	2053	8148
1967	2363	2108	2108	8203
1972	2391	2112	2112	8207
1987	2397	2152	2152	8247
1992	2443	2193	2193	8288
2012	2487	2207	2207	8302
2030	2497	2222	2222	8317
2031	2508	2327	2327	8422
2053	2516	2363	2363	8458
2082	2520	2391	2391	8486
2097	2547	2397	2397	8492
2108	2553	2443	2443	8538
2112	2557	2487	2487	8582
2128	2572	2508	2508	8603
2142		2516	2516	8611
		2520	2520	8615
		2553	2553	8648
		2557	2557	8652
		2572	2572	8667

*8107 is an actual exception; the other integers in E are not.

Case XIII. PROOF THAT ALL INTEGERS FROM 1775 + 6924 =
8699 TO 2606 + 6924 = 9530 ARE SUMS OF 5 VALUES OF $f(x)$.

-901	-901	-901	-1877	+829	E
1793	2002	2302	1793		
1798	2003	2303	1798		
1804	2019	2327	1804		
1823	2030	2328	1823		
1831	2038	2333	1831		
1849	2039	2335	1849	1849	8773
1853	2048	2370	1853	1853	8777
1857	2049	2374	1857		
1863	2068	2383	1863		
1865	2104	2398	1865	1865	8789
1869	2148	2399	1869	1869	8793
1903	2153	2408	1929	1929	8853
1907	2192	2422	1934	1934	8858
1929	2198	2437	1939	1939	8863
1934	2199	2449	1968	1968	8892
1939	2204	2463	1974	1974	8898
1953	2214	2469	2002	2002	8926
1959	2218	2473	2030	2030	8954
1963	2219	2479	2039	2039	8963
1967	2223	2497	2048	2048	8972
1968	2224	2528	2049	2049	8973
1972	2238	2539	2192	2192	9116
1973	2242	2547	2199	2199	9123
1974	2274	2559	2214	2214	9138
1978	2289	2573	2219	2219	9143
1992	2294	2590	2223	2223	9147
1999			2224	2224	9148
			2238	2238	9162
			2294	2294	9218
			2302	2302	9226
			2303	2303	9227
			2333	2333	9257
			2335	2335	9259
			2370	2370	9294
			2408	2408	9332
			2479	2479	9403
			2497	2497	9421
			2539	2539	9463
			2590	2590	9514

Although 89 apparent exceptions were introduced in Cases VI to XIII above, only 4 integers in the interval 2734 to 9530 require more than 5 values, namely, 2978, 3353, 3372 and 8107. Each one, however, can be expressed as a sum of 6 values. A possible set of decompositions is:

$$2978 = 1 + 1 + 1 + 175 + 1105 + 1695$$

$$3353 = 1 + 1 + 15 + 175 + 1105 + 2056$$

$$3372 = 1 + 15 + 260 + 369 + 671 + 2056$$

$$8107 = 1 + 1 + 5 + 505 + 671 + 6924$$

With these four exceptions, all integers from 2734 to 9530 have been proved by the work of Cases I to XIII to be sums of 5 values of $f(x) = (x^3 + x)/2$.

7. Proof that All Integers from 2734 to 97,079 Are Sums of 6 Functional Values. Since 0 is a functional value, it follows from Section 6 that all the integers from 2734 to 9530 are sums of 6 values. With the interval 4622-9243, which contains only one integer requiring 5 values, as a basis, the methods of the previous section have been applied to prove that all integers from 9263 to 97,079 are sums of 6 values. No exceptions occur.

In the construction of the proof, the work is divided into 36 subdivisions. For these the needed functional values and differences are displayed here:

(a) Values of $f(x) = (x^3 + x)/2$ from $f(21) = 4641$
to $f(56) = 87,836$, and Their Needed Differences

$f(x)$	Difference	$f(x)$	Difference
4641	694	29,679	2341
5335	760	32,020	2461
6095	829	34,481	2584
6924	901	37,065	2710
7825	976	39,775	2839
8801	1054	42,614	2971
9855	1135	45,585	3106
10,990	1219	48,691	3244
12,209	1306	51,935	3385
13,515	1396	55,320	3529
14,911	1489	58,849	3676
16,400	1585	62,525	3826
17,985	1684	66,351	3979
19,669	1786	70,330	4135
21,455	1891	74,465	4294
23,346	1999	78,759	4456
25,345	2110	83,215	4621
27,455	2224	87,836	
29,679			

(b) Proof for 36 Subdivisions of the Interval

9263 to 97,079

In column A are entered the boundaries of the interval 4622-9243, and the integer 8107, the only one in this interval which requires 5 values. The same A is used for each of the 36 subdivisions of the proof. The entries in the columns headed, "+ a functional value," show in each case the boundaries of the interval which is proved to contain only integers that are expressible as a sum of 6 values.

A	-694	+4641	-760	+5335	+829	+6095
4622 8107 9243	7413	9263 13,884	7347	9957 14,578	7278	10,717 15,338
A	-901	+6924	-976	+7825	-1054	+8801
4622 8107 9243	7206	11,546 16,167	7131	12,447 17,068	7053	13,423 18,044
A	-1135	+9855	-1219	+10,990	-1306	+12,209
4622 8107 9243	6972	14,477 19,098	6888	15,612 20,233	6801	16,831 21,452
A	-1396	+13,515	-1489	+14,911	-1585	+16,400
4622 8107 9243	6711	18,137 22,758	6618	19,533 24,154	6522	21,022 25,643
A	-1684	+17,985	-1786	+19,669	-1891	+21,455
4622 8107 9243	6423	22,607 27,228	6321	24,291 28,912	6216	26,077 30,698

(b) Proof for 36 Subdivisions of the Interval

9263 to 97,079 (Continued)

A	-1999	+23,346	-2110	+25,345	-2224	+27,455
4622	6108	27,968	5997	29,967	5883	32,077
8107		32,589		34,588		36,698
9243						
A	-2341	+29,679	-2461	+32,020	-2584	+34,481
4622	5766	34,301	5646	36,642	5523	39,103
8107		38,922		41,263		43,724
9243						
A	-2710	+37,065	-2839	+39,775	-2971	+42,614
4622	5397	41,687	5268	44,397	5136	47,236
8107		46,308		49,018		51,857
9243						
A	-3106	+45,585	-3244	+48,691	-3385	+51,935
4622	5001	50,207	4863	53,313	4722	56,557
8107		54,828		57,934		61,178
9243						
A	-3529	+55,320	-3676	+58,849	-3826	+62,525
4622	4578	59,942	4431	63,471	4281	67,147
8107		64,563		68,092		71,768
9243						
A	-3979	+66,351	-4135	+70,330	-4294	+74,465
4622	4128	70,973	3972	74,952	3813	79,087
8107		75,594		79,573		83,708
9243						
A	-4456	+78,759	-4621	+83,215		+87,836
4622	3651	83,381	3486	87,837		92,458
8107		88,002		92,458		95,943
9243						97,079

None of the remainder entries occurring in the preceding proof requires more than 5 functional values, and none of the final entries requires more than 6 functional values. In the last column,

$$95,943 = 34 + 34 + 65 + 260 + 2925 + 92,625,$$

a sum of 6 values.

Thus the proof is completed that all integers from 2734 to 97,079 are sums of 6 values of $f(x) = (x^3 + x)/2$.

(c) Generalization of the Extension Process

In general, denote the lower and upper limits of a range of integers missing from a table of sums of $k - 1$ values, by a and b respectively. Denote the largest function to be added by $f(m + 1)$. Then the validity of the extension process employed in Sections 6 and 7 depends upon

$$(26) \quad f(m + 1) - f(m) \leq a.$$

In addition, assurance of overlapping between the extended portions requires that

$$(27) \quad a + f(m + 1) \leq b + f(m).$$

Finally, when ascent is made from a table of extensions based upon a list of integers requiring sums of $k - 1$ values in the range a to b , to a table of extensions based upon a list of integers requiring sums of k values in the range A to B , it is necessary that

$$(28) \quad b + f(m + 1) \geq B.$$

Conditions (26) to (28) are satisfied by the quantities of Sections 6 and 7, thus:

$$\text{When } a = 1775, b = 2606,$$

$$f(m + 1) = f(24) = 6924, \quad f(23) = 6095,$$

equation (26) becomes

$$6924 - 6095 < 1775;$$

equation (27) becomes

$$1775 + 6924 < 2606 + 6095;$$

equation (28) becomes

$$2606 + 6924 > 9243.$$

Also, when $A = 4622$, $B = 9243$,

$$f(M + 1) = f(56) = 87,836, \quad f(55) = 83,215,$$

equation (26) becomes

$$87,836 - 83,215 < 4622;$$

equation (27) becomes

$$4,622 + 87,836 = 9243 + 83,215;$$

equation (28) becomes

$$9,243 + 87,836 = 97,079 > g = 96,458$$

in Section 8.

8. Conclusion: Universal Theorem II. Stated in the paper of Dickson previously cited is this theorem of ascent:

THEOREM 3. Let a polynomial $f(x)$ take integral values ≥ 0 for all integers $x \geq 0$; let $f(x + 1) - f(x)$ increase with x . Suppose that every integer n for which $c < n \leq g$ is a sum of $k - 1$ values of $f(x)$ for integers $x \geq 0$. Let m be the maximum

integer for which $f(m+1) - f(m) < g - c$. Then every integer N
for which $c + f(0) < N \leq g + f(m+1)$ is a sum of k values of
 $f(x)$ for integers $x \geq 0$.

Theorem 3 may now be applied to $f(x) = (x^3 + x)/2$, since $f(x+1) - f(x) = (3x^2 + 3x + 2)/2$ increases with x . When m is the maximum integer for which $f(m+1) - f(m) < g - c$, this inequality may be written:

$$(29) \quad 3(2m+1)^2 < 8(g-c) - 5.$$

From the results of Sections 5 and 7, every integer n for which $63 < n \leq 97,079$ is a sum of 6 values of $f(x)$. Actually $g = 96,458$ suffices. The maximum integer m for which

$$3(2m+1)^2 < 8(96,458 - 63) - 5$$

has $2m+1 = 507$, $m = 253$. Since $f(254) = 8,193,659$, Theorem 3 shows that all integers from 64 to 8,290,117 are sums of 7 values of $f(x)$. The same is true of 29, 58 and 63. Hence Theorem 3 may be applied with $c = 0$, $g = 8,290,117$, $k = 8$; the maximum m is 2350, and all integers $\leq 6,505,516,068$ are sums of 8 values of $f(x)$. The next maximum m is 65,855, and all integers $\leq 142,815,660,920,004$ are sums of 9 values of $f(x)$. This limit exceeds $504 \cdot 3^{24} = 142,344,486,386,424$, hence the proof is completed for

THEOREM II. Every integer ≥ 0 is a sum of 9 values of
 $(x^3 + x)/2$ for integers $x \geq 0$.

VITA

Frances Ellen Baker was born December 19, 1902, at Anna, Illinois. In 1919 she graduated from the public high school of Iowa City, Iowa. She received the Bachelor of Arts degree from the State University of Iowa in June, 1923, and the Master of Science degree from the same institution in June, 1925. Holding a graduate scholarship at the University of Iowa in 1924-25, she taught a freshman mathematics course for one semester.

From 1925 to 1927 she was head of the department of mathematics and physics at Tabor College, Tabor, Iowa, and during 1927-28 held a similar position in the Junior College of Jefferson City, Missouri. She then studied mathematics at the University of Chicago through two quarters of 1929. After an interval of two years during which she was instructor in mathematics and head of the department in the Creston, Iowa, Junior College and Senior High School, she returned in 1931 to the University of Chicago. There during 1932-33 she was a Fellow of the Department of Mathematics and taught a freshman course in the Spring Quarter.

At the University of Chicago her instructors have been Professors Albert, Barnard, Bartky, Bliss, Dickson, Lane, Logsdon, Lunn, MacMillan and Slaught. To all these, and especially to Professor Dickson, under whose encouraging guidance this thesis has been written, she wishes to acknowledge her indebtedness and deep appreciation.

